1 Punchline of the talk

Every Poisson manifold is naturally partitioned into regularly immersed symplectic manifolds called symplectic leaves.

2 Symplectic Manifolds

**Defn 1.** A differential two-from $\omega$ on a differentiable manifold $M$, is non-degenerate at a point $p \in M$ if $\omega(u_p, v_p) = 0$ for all $v_p$ implies $u_p = 0$ for all tangent vectors $u_p, v_p \in T_pM$. Furthermore, we say that a two-form is non-degenerate if it is non-degenerate at every point on the manifold.

**Defn 2.** A Symplectic Manifold $(M, \omega)$ is a differentiable manifold $M$ with a distinguished non-degenerate two-form $\omega$ called the Symplectic Form.

We find that all symplectic manifolds are necessarily of even dimension.

**Defn 3.** A Symplectomorphism $\phi$ between two symplectic manifolds $(X, \omega_X)$ and $(Y, \omega_Y)$ is a diffeomorphism $\phi : X \to Y$ such that $\phi^*\omega_Y = \omega_X$.

The most basic example of a symplectic manifold is the plane $\mathbb{R}^2$ equipped with the two-form $\omega = dx \wedge dy$. This can be generalized to any even dimensional real space $\mathbb{R}^{2n}$. If we take coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$ then we define our two-form to be $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$.

We may also define a symplectic structure on the cotangent bundle $T^*M$ of any differentiable manifold $M$.

**Thm 1.** (Darboux) Every $2n$-dimensional symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega)$, with $\omega$ as above.

Any differentiable function $h : M \to \mathbb{R}$ on a symplectic manifold $(M, \omega)$, uniquely determines a vector field $X_h$ on $M$. This is called the Hamiltonian Vector Field of $h$.

3 Poisson Manifolds

**Defn 4.** A Poisson manifold $P$ is a differentiable manifold equipped with a Lie algebra structure on $C^\infty(P)$ defined by a Poisson Bracket 

{·,·} : $C^\infty(P) \times C^\infty(P) \rightarrow C^\infty(P)$

satisfying the following conditions:

- $\{\alpha f + \beta g, h\} = \alpha \{f, h\} + \beta \{g, h\}$
- $\{f, g\} = -\{g, f\}$
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
- $\{fg, h\} = f\{g, h\} + g\{f, h\}$

for all $f, g, h \in C^\infty(P)$ and $\alpha, \beta \in \mathbb{R}$ and with each condition holding for both entries.

We can equip any differentiable manifold $M$ with a trivial Poisson structure simply by setting $\{f, g\} = 0$ for all $f, g \in C^\infty(M)$.

We can equip any symplectic manifold $(M, \omega)$ with a Poisson bracket $\{·,·\}_\omega$ by using the Hamiltonian vector fields of functions in $C^\infty(P)$.

$$\{f, g\}_\omega = \omega(X_f, X_g)$$

For any Poisson manifold $P$, by fixing some $h \in C^\infty(P)$ we can define a linear map $X_h : C^\infty(P) \rightarrow C^\infty(P)$.

$$X_h = \{\cdot, h\} : C^\infty(P) \rightarrow C^\infty(P)$$

Since our Poisson bracket satisfies Leibniz’s rule by definition, it follows that $X_h$ defines a vector field on the Poisson manifold $P$. This is a generalized form of the Hamiltonian Vector Field for all Poisson manifolds and it agrees with the definition for symplectic Poisson manifolds.
4 The Splitting Theorem

**Defn 5.** A Poisson mapping is a smooth mapping 
\[ \phi : P_1 \to P_2 \] 
between two Poisson manifolds \( P_1 \) and \( P_2 \) with Poisson brackets \( \{\cdot,\cdot\}_1 \) and \( \{\cdot,\cdot\}_2 \) respectively, such that 
\[ \{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi \] 
for all \( f, g \in C^\infty(P_2) \).

**Defn 6.** Given two Poisson manifolds \( P_1 \) and \( P_2 \) we define the product \( P_1 \times P_2 \) to be the usual product of smooth manifolds equipped with a Poisson bracket \( \{\cdot,\cdot\} \) such that each of the projections \( \pi_i : P_1 \times P_2 \to P_i \) are Poisson mappings and 
\[ \{f \circ \pi_1, g \circ \pi_2\} = 0 \] 
for all \( f \in C^\infty(P_1) \) and \( g \in C^\infty(P_2) \).

**Thm 2.** (Splitting Theorem) Let \( P \) be a Poisson manifold, and let \( p \in P \). Then there exists an open neighbourhood \( U \subset P \) containing \( p \) and a unique diffeomorphic Poisson mapping 
\[ \phi = \phi_S \times \phi_N : U \to S \times N \] 
where \( S \) is a symplectic manifold (called a Symplectic Leaf) and \( N \) is a Poisson manifold.

In other words, every Poisson manifold is naturally partitioned into immersed symplectic manifolds.

5 Examples

**Ex 1.** On a differentiable manifold with the trivial Poisson structure, its symplectic leaves will be exactly the individual points \( x \in M \).

**Ex 2.** If we take any symplectic manifold \((M, \omega)\) and equip it with the Poisson bracket \( \{\cdot,\cdot\}_\omega \) then \( M \) will have a single symplectic leaf; itself.

**Ex 3.** Take any symplectic manifold \( M \) equipped with the symplectic Poisson structure, and any differentiable manifold \( N \) equipped with the trivial Poisson structure. Then \( M \times N \) will be a Poisson manifold with its symplectic leaves being \( \{M \times \{x\} | x \in N\} \).

**Ex 4.** The dual \( g^* \) of a Lie algebra \((g, [\cdot,\cdot])\) is a Poisson manifold, where \([\cdot,\cdot]\) is the standard Lie bracket given by 
\[ [X,Y] = XY - YX \] 
for \( X, Y \in g \). As \( C^\infty(g^*) = (g^*)^* = g \), we can simply define a Poisson bracket \( \{X,Y\} = [X,Y] \) for any \( X, Y \in g \). In this case the symplectic leaves are realized as the coadjoint orbits of the Lie group \( G \).

Note, the example 3 and example 4 may give us nonsymplectic manifolds.

6 References

- A. Zamorzaev, *Introduction to Symplectic Topology*, Lecture Notes