Introduction to Immersion, Embedding, and the Whitney Embedding Theorems

Paul Rapoport
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Immersion

Let $M, N$ be differentiable manifolds. Then an immersion is a differentiable function $f : M \to N$ whose derivative is everywhere injective. Alternatively, an immersion is a map that is injective on the tangent spaces, that is, $D_pf : T_pM \to T_{f(p)}N$ for all points $p \in M$.

![Figure 1: An immersion of an open interval into $\mathbb{R}^2$.](image1)

All crossings must be transverse! Any non-transverse crossing makes the map fail to be an immersion, since at the nontransverse intersection, the tangent spaces fail to be distinct.

Embedding

An embedding on a compact manifold is just an immersion that is also injective. In essence we’re finding a submanifold of the target space that looks like our domain manifold. Shouldn’t we always
be able to do this? We need to prove it!

The Whitney Embedding Theorem

We start by proving that there really is some $N$ for which we can embed $f : M \to \mathbb{R}^N$. We use bump functions and a partition of unity subordinate to a finite subcover. We can show that the result of this is injective, and so is the differential.

The maps from the elements of the subcover:

$$\tilde{\phi}_i(x) = \begin{cases} \phi_i(x)f_i(x), & x \in V_i \\ 0, & x \notin V_i. \end{cases}$$  \hfill (1)

Pasting the maps together:

$$\Phi(x) : M \to \mathbb{R}^{k(dim M + 1)}, \ x \mapsto (\tilde{\phi}_1(x), \tilde{\phi}_2(x),..., \tilde{\phi}_k(x), f_1(x), f_2(x),..., f_k(x)).$$  \hfill (2)

with differential

$$D\Phi(x) = Df_1(x)\phi_1(x) + D\phi_1(x)f_1(x),..., Df_k(x)\phi_k(x) + D\phi_k(x)f_k(x), Df_1(x),..., Df_k(x).$$  \hfill (3)

We can do better! Can we find reasonable bounds for $N$? We can easily show that $N \geq 2 \dim(M) + 1$ suffices. We prove this by picking an embedding in some high dimension, projecting down to a hyperplane of some smaller dimension, and showing that the set of vectors on which the composition of easy embedding and easy projection doesn’t work is measure 0.

$$g(x_1, x_2) = \frac{\Phi(x_2) - \Phi(x_1)}{||\Phi(x_2) - \Phi(x_1)||},$$  \hfill (4)

$$D(\Phi \circ \phi^{-1}) : U \times S^{n-1} \to S^{N-1},$$  \hfill (5)

Can we still do better? Yes! We can show that $N \geq 2 \dim(M)$ is a sharp boundary, though this is a lot trickier. If the dimension is large, we can create new transverse intersections and then remove them in pairs. If the dimension is small, we just check by hand, using classification theorems for manifolds of dimension 2 or less, which is easy.