Symplectic and Poisson Manifolds

Harry Smith

In this survey we look at the basic definitions relating to symplectic manifolds and Poisson manifolds and consider different examples of these. We go on to see how symplectic and Poisson manifolds are related through the splitting theorem.

1 Introduction

Symplectic and Poisson manifolds both arise from classical mechanics. Symplectic manifolds are essentially a generalization of the phase space of a closed system. As we will see symplectic manifolds come equipped with a natural vector field called the *Hamiltonian vector field* that corresponds to the Hamiltonian equations of a system.

Symplectic manifolds are a special case of Poisson manifolds, and as we will state in our main theorem, all Poisson manifolds naturally partition into symplectic manifolds (its *leaves*). Indeed, a motivation for studying Poisson manifolds is that frequently it can be simpler to study a Poisson manifold that a symplectic manifold is embedded into, than it is to study a symplectic manifold itself. For example, it is often easier to analyze stability of the dynamics on a Poisson manifold than on its symplectic leaves.

2 Symplectic Manifolds

Let $\omega \in \bigwedge^2 T^*M$ be a two-form on a differentiable manifold M. So

$$\omega = \sum_{i<j} \omega_{ij}(x) dx_i \wedge dx_j$$

where $\omega_{ij} \in C^\infty(M)$.

We may view ω as a bilinear form on the tangent space of M.

$$\omega_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$

We say ω is *non-degenerate* at a point $p \in M$ if $\omega(u_p, v_p) = 0$ for all v_p implies $u_p = 0$ for all tangent vectors $u_p, v_p \in T_pM$. Furthermore, we say that a two-form is non-degenerate if it is non-degenerate at every point on the manifold.

Definition .1. A *Symplectic Manifold* (M, ω) is a differentiable manifold M with a distinguished non-degenerate two-form ω called the *Symplectic Form.*

We find that all symplectic manifolds are necessarily of even dimension.

Definition .2. A *Symplectomorphism* ϕ between two symplectic manifolds (X, ω_X) and (Y, ω_Y) is a diffeomorphism $\phi : X \to Y$ such that $\phi^* \omega_Y = \omega_X$.

Example 2.1. The most basic example of a symplectic manifold is the plane $\mathbb{R}^2$ equipped with the two-form $\omega = dx \wedge dy$. This can be generalized to any even dimensional real space $\mathbb{R}^{2n}$. If we take coordinates $(x_1, ..., x_n, y_1, ..., y_n) \in \mathbb{R}^{2n}$ then we define our two-form to be

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

and this will be non-degenerate.

A theorem of Darboux below shows us that the above example of symplectic manifold should be considered as our standard symplectic real space.

Theorem .3. Every 2n-dimensional symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega)$, with ω as above.

Example 2.2. We know that any n-dimensional oriented manifold will have a non-degenerate n-form, so it follows that any 2-dimensional oriented manifold has a non-degenerate two-form and hence will be symplectic.

Example 2.3. We can define a symplectic structure on the cotangent bundle T^*M of any differentiable manifold M. We construct our distinguished two-form ω as follows. Take the projection $\pi : T^*M \to M$ and consider it's pushforward

$$\pi_* : T(T^*M) \longrightarrow TM$$

Now if we take a tangent vector of T^*M

$$v_{\alpha_x} \in T_{\alpha_x}(T_x^*M)$$

at the cotangent vector $\alpha_x \in T_x^*M$. Then define a one-form $\lambda \in T^*(T^*M)$ by

$$\begin{array}{rccc} \lambda : & T_{\alpha_x}(T_x^*M) & \longrightarrow & \mathbb{R} \\ & v_{\alpha_x} & \longmapsto & \alpha_x(\pi_* v_{\alpha_x}) \end{array}$$

We then set our non-degenerate two-form to be $\omega = d\lambda$. We actually find that the symplectic structure on $\mathbb{R}^{2n}$ above is the same as the symplectic structure defined here on $T^*(\mathbb{R}^n)$.

As mentioned in the introduction, every symplectic manifold comes with a natural vector field. We construct this now. First note that, on any symplectic manifold (M, ω) , since the symplectic form ω is non-degenerate, it sets up an isomorphism

$$\omega : TM \longrightarrow T^*M$$

With inverse

$$\omega^{-1} : T^*M \longrightarrow TM$$

Hence, we may identify any one-form on a symplectic manifold with a vector field. Furthermore, any differentiable function $h : M \to \mathbb{R}$ uniquely determines a vector field X_h. We call this vector field the *Hamiltonian Vector Field*.

3 Poisson Manifolds

Definition .4. A *Poisson manifold* P is a differentiable manifold equipped with a Lie algebra structure on $C^\infty(P)$ defined by a *Poisson Bracket*

$$\{\cdot, \cdot\} : C^\infty(P) \times C^\infty(P) \longrightarrow C^\infty(P)$$

satisfying the following conditions:

- *Bilinearity*: $\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$
- *Skew symmetry*: $\{f, g\} = -\{g, f\}$
- *Jacobi identity*: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
- *Leibniz's rule*: $\{fg, h\} = f\{g, h\} + g\{f, h\}$

for all $f, g, h \in C^\infty(P)$ and $\alpha, \beta \in \mathbb{R}$ and with each condition holding for both entries.

We will now consider some of the more basic examples of Poisson manifolds.

Example 3.1. We can equip any differentiable manifold M with a trivial Poisson structure simply by setting $\{f, g\} = 0$ for all $f, g \in C^\infty(M)$.

Example 3.2. Recall how any smooth function $h : M \to \mathbb{R}$ on a symplectic manifold (M, ω) uniquely determines the *Hamiltonian Vector Field* X_h. We will now use the Hamiltonian vector field to show how we may equip any symplectic manifold with a natural Poisson bracket $\{\cdot, \cdot\}_\omega$. We do this as so

$$\{f, g\}_\omega = \omega(X_f, X_g)$$

This is clearly bilinear by the bilinearity of two-forms. Furthermore

$$\{f, g\}_\omega = \omega(X_f, X_g) = -\omega(X_g, X_f) = -\{g, f\}$$

Also, as

$$X_{fg} = \{\cdot, fg\} = f\{\cdot, g\} + g\{\cdot, f\} = fX_g + gX_f$$

It follows that

$$\begin{aligned}\{fg, h\}_\omega &= \omega(X_{fg}, X_h) = \omega(fX_g + gX_f, X_h)\\ &= f\omega(X_g, X_h) + g\omega(X_f, X_h)\\ &= f\{g, h\} + g\{f, h\}\end{aligned}$$

Hence our bracket also satisfies the Leibniz property.

Now that we have seen how the Hamiltonian vector fields of a symplectic manifold may be used to turn any symplectic manifold into a Poisson we may introduce a more generalised notion of the Hamiltonian vector field for all Poisson manifolds. We do this now.

By fixing some $h \in C^\infty(P)$ we can define a linear map

$$X_h = \{\cdot, h\} : C^\infty(P) \longrightarrow C^\infty(P)$$

Now, since our Poisson bracket satisfies Leibniz's rule by definition, it follows that X_h defines a vector field on the Poisson manifold P. This is the Hamiltonian vector field for Poisson manifolds. We find that this definition of the Hamiltonian vector field agrees with the earlier definition for symplectic manifolds if we consider the natural Poisson structure on symplectic manifolds discussed above. This is true almost by definition.

Furthermore we can use the Hamiltonian vector field of Poisson manifolds to define a natural bundle map from the cotangent space of P at p, to the tangent space of P at p. First we define a define a tangent vector $X_h|_p$ by

$$\begin{array}{rccc} X_h|_p : & C^\infty(P) & \longrightarrow & \mathbb{R} \\ & f & \longmapsto & X_H(f)(p) \end{array}$$

Now we define our bundle map as so

$$\begin{array}{rccc} B_p : & T_p^*M & \longrightarrow & T_pM \\ & (df)_p & \longmapsto & X_f|_p \end{array}$$

We now only require a few very natural definitions before we can state our main result.

Definition .5. The *rank* of a Poisson manifold P at a point $p \in P$ is defined to be the rank of the map B_p.

Definition .6. A *Poisson mapping* is a smooth mapping $\phi : P_1 \to P_2$ between two Poisson manifolds P_1 and P_2 with Poisson brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ respectively, such that

$$\{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi$$

for all $f, g \in C^\infty(P_2)$.

Definition .7. Given two Poisson manifolds P_1 and P_2 we define the product $P_1 \times P_2$ to be the usual product of smooth manifolds equipped with a Poisson bracket $\{\cdot,\cdot\}$ such that each of the projections $\pi_i : P_1 \times P_2 \to P_i$ are Poisson mappings and $\{f \circ \pi_1, g \circ \pi_2\} = 0$ for all $f \in C^\infty(P_1)$ and $g \in C^\infty(P_2)$.

We are now ready to state the *Splitting Theorem* for Poisson manifolds.

Theorem .8. Let P be a Poisson manifold, and let $p \in P$. Then there exists an open neighbourhood $U \subset P$ containing p and a unique diffeomorphic Poisson mapping

$$\phi = \phi_S \times \phi_N : U \longrightarrow S \times N$$

where S is a symplectic manifold and N is a Poisson manifold with rank zero at $\phi_N(p)$.

S in the above theorem is called a *symplectic leaf* of P. In other words, this theorem states that every Poisson manifold is naturally partitioned into regularly immersed symplectic leaves.

Looking back, we can see how the splitting theorem applies to our previous examples. On a differentiable manifold with the trivial Poisson structure, the rank of M will be 0 everywhere, and its symplectic leaves will be exactly the individual points $x \in M$. Furthermore, if we take any symplectic manifold (M, ω) and equip it with the Poisson bracket $\{\cdot,\cdot\}_\omega$ then M will have a single symplectic leaf; itself. We will now look at some more examples of Poisson manifolds and their symplectic leaves.

Example 3.3. Take any symplectic manifold M equipped with the symplectic Poisson structure, and any differentiable manifold N equipped with the trivial Poisson structure. Then $M \times N$ will be a Poisson manifold with its symplectic leaves being $\{M \times \{x\} | x \in N\}$.

Example 3.4. The dual $\mathfrak{g}^*$ of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is a Poisson manifold, where $[\cdot,\cdot]$ is the standard Lie bracket given by

$$[X, Y] = XY - YX$$

for $X, Y \in \mathfrak{g}$. As $C^\infty(\mathfrak{g}^*) = (\mathfrak{g}^*)^* = \mathfrak{g}$, we can simply define a Poisson bracket $\{X, Y\} = [X, Y]$ for any $X, Y \in \mathfrak{g}$. In this case the symplectic leaves are realized as the coadjoint orbits of the Lie group G.

Note, the above two examples may give us non-symplectic manifolds.

4 References

1. A. Cannas da Silva, *Symplectic Geometry*, Handbook of Differential Geometry, 2004

2. J. Watts, *An Introduction to Poisson Manifolds*, 2007
3. A. Weinstein, *The Local Structure of Poisson Manifolds*, J. Differential Geometry, 1983
4. A. Zamorzaev, *Introduction to Symplectic Topology*, Lecture Notes