An Introduction to Transversality

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Abstract

This is intended as an introduction to the basic concepts of transversality of two manifolds, and of maps with respect to manifolds. I will introduce the concepts of *stability* and *genericity* of a property, and prove that transversality is both stable and generic.

1 Introduction

Transversality is a description of how two objects intersect. As I will explain throughout the course of this survey it can be thought of as the "opposite" of tangency, in a number of ways. For two submanifolds to be transverse, their tangent spaces at all points of intersection must span the tangent space of the ambient manifold. In particular, transversality always fails whenever two submanifolds are tangent. But, more importantly, tangency is not a stable property: any situation where two objects are tangent can be easily and slightly perturbed into a non-tangent situation. As I will show, the opposite is true of transversality. Transversality *is* stable, which is part of what makes it such a powerful tool.

The concept of transversality was developed by the French mathematician Rene Thom in the 1950s. His 1954 doctoral thesis, "possibly the best math PhD thesis of all time" [1], included the statement and proof of what later became known as Thom's Transversality Theorem. I provide a proof in 2.5. In fact, this Theorem shows that transversality is *generic*, meaning that any non-transverse intersection can deformed (by an arbitrarily small deformation) into a transverse intersection. This is an even stronger property than stability.

As we will see, transversality is quite useful for extending the Preimage Theorem to say when the preimage of a manifold (and not just a single point) under a smooth map is also a manifold. In addition, that transversality is both stable and generic make it a powerful criterion. As we shall see in Section 3, proving a statement for the transversal case shows that the statement is *almost always* true.

2 Theory of Transversality

2.1 Terminology

We begin with a simple definition of what it means for two vector spaces to be transverse. This definition extends naturally to intersections of submanifolds, or intersections of maps and manifolds, by considering the submanfolds' tangent spaces as vector spaces.

Definition 1. *Let F and G be vector subspaces of a vector space E. F and G are* ***transverse*** *if*

$$F + G = E.$$

In this case, we write $F \pitchfork G$.

Note that the success or failure of transversality depends on the dimensions of F, G and E. If

$$\dim F + \dim G < \dim E,$$

then F and G cannot be transverse. We should keep in mind, in all that follows, that transversality depends not just on having a nice intersection (of two vector spaces, two manifolds, etc...) but on the space in which they are embedded. In this case, we can always cause transversality to fail by consider F and G as vector subspaces of a vector space of sufficiently large dimension.

The notion of transversality can easily be extended to two submanifolds of a given manifold.

Definition 2. *Let M, N be submanifolds of a manifold Y. M and N are* ***transverse****, (or, equivalently, M and N* ***intersect transversally****) if, at every point $x \in M \cap N$,*

$$T_x M + T_x N = T_x Y$$

Again, we write $M \pitchfork N$.

Considering the tangent spaces of M and N as vector subspaces of the tangent space of Y, we can say that $M \pitchfork N$ are transverse if their tangent spaces, considered at every point of intersection, are transverse in the tangent space of Y.

It is important to note that this is a condition on points in the intersection of M and N. Therefore, whenever $M \cap N = \varnothing$ the condition is (trivially) satisfied and $M \pitchfork N$.

Again, as with vector spaces, whenever M and N intersect nontrivially, transversality requires $\dim M + \dim N \geq \dim Y$. Thus transversality again depends on the choice of the ambient manifold Y.

In order to extend our definition to the intersection of a smooth map f with a manifold, we will need to associate a vector space to f. The derivative of f supplies just the vector space we need, since at any point $a \in X$, $D_a f$ is a map from the tangent space $T_a X$ to the tangent space $T_{f(a)} Y$. We take $D_a f(T_a X)$ to be the full image of $T_a X$ under the map $D_a f$.

Definition 3. *Let $f : X \to Y$ be a smooth map and Z a submanifold of Y. We say that f is* ***transverse*** *to Z if, for every $a \in f^{-1}(Z)$,*

$$D_a f(T_a X) + T_{f(a)} Z = T_{f(a)} Y$$

And we write $f \pitchfork Z$.

This says that a map f and a submanifold Z are transverse when the image of the tangent space of X, under the derivative map induced by f, is transverse to the tangent space of Z, when both are considered as submanifolds of Y. Then, much as for the intersection of two manifolds, we observe:

- If $f^{-1}(Z)$ is empty, then f and Z are (trivially) transverse.
- If $f(X)$ and Z do intersect, then transversality will automatically fail if

$$\text{rank } D_a f + \dim T_{f(a)} Z < \dim T_{f(a)} Y$$

  for any $a \in f^{-1}(Z)$.

In effect, we have defined the transversality of a map f with a submanifold by considering the image of f as a submanifold, and then applying our criteria for the transversality of two submanifolds. In the same way, we would consider two smooth maps f and g to be transverse if their images, when considered as submanifolds of a common ambient manifold, are transverse.

2.2 Generalization of the Preimage Theorem.

Transversal intersections are very nice: the intersections themselves are submanifolds.

The notion of transversality allows us to generalize the Preimage Theorem, and determine when the preimage of a manifold (and not just a single point) under a smooth map is also a manifold. Recall:

Theorem 4 (The Preimage Theorem)**.** *Let $F : M \to N$ be a smooth map between manifolds, and let $c \in N$ such that at each point $a \in F^{-1}(c)$, the derivative DF_a is surjective. Then $F^{-1}(c)$ is a smooth manifold of dimension dim M - dim N.*

We refer the reader to Theorem 3.3 in Hitchin's notes [4] for a proof of the Preimage Theorem. Now, let's return to our map $f : X \to Y$, where X and Y are manifolds, and Z is a submanifold of Y. Recall that the *codimension* of Z in Y is the difference of their dimensions.

Theorem 5. *If $f \pitchfork Z$, then $f^{-1}(Z)$ is a manifold. Moreover, the codimension of Z in Y is equal to the codimension of $f^{-1}(Z)$ in X.*

Proof. We will construct a smooth map $F : Y \to \mathbb{R}$, where $F^{-1}(0) = f^{-1}(Z)$, and then apply the Preimage Theorem. [3]

Let $a \in Z$. Since Z is a submanifold of Y, we can choose a coordinate chart in a neighborhood around a such that

$$\varphi = (c_1, \ldots, c_z, c_{z+1}, \ldots, c_y),$$

where where y and z are the dimensions of Y and Z, and the first z coordinates of φ are the coordinates of Z.

Define

$$\begin{array}{rccl} g: & Y & \to & \mathbb{R}^{y-z} \\ & \tilde{y} & \mapsto & c_{z+1}(\tilde{y}), \ldots, c_y(\tilde{y}) \end{array}$$

Then, in the neighborhood of a where φ is defined, g is a smooth function and

$$Z = g^{-1}(0)$$

and $D_z g$ is surjective $\forall g \in f^{-1}(0)$.

Consider the map $F = g \circ f : X \to \mathbf{R}^{y-z}$. It is a composition of smooth maps and therefore smooth, and $F^{-1}(0) = f^{-1}(Z)$.

Now, to show that $f^{-1}(Z)$ is a manifold, we must show that $D_a F$ is surjective $\forall a \in F^{-1}(0)$.

But

$$D_a F = D_{f(a)} g \cdot D_a f.$$

Thus $D_a F$ will be surjection onto R^{y-z} whenever $D_a f$ is a surjection onto R^{y-z}. Happily, that is exactly what our transversality condition tells us. □

It follows that:

Proposition 6. *If $M \pitchfork N$ in some ambient manifold Y, then their intersection is a manifold.*

Proof. Consider the natural immersion map $i : M \to Y$. Then $i(M) = M$, and $i \pitchfork Y$.

□

2.3 Stability of a Property.

A property of a map is said to be stable if it cannot be "undone" by small deformations of the map. We have all spent years working with stable properties, and using the fact of their stability, without ever using the term. Consider differentiability: if a (continuous) function is differentiable, then we can deform it a bit in any direction we like, and the resulting function will still be differentiable.

The property of transversality (of a map with respect to a given submanifold Z) is also stable. This is another way in which transversality is the "opposite" of tangency, which can be undone with arbitrary small deformation.

We formalize the concept of stability using homotopy families.

Recall that,

Definition 7. *Two maps $f_0 : X \to Y$ and $f_1 : X \to Y$ are said to be **homotopic** if there exists a smooth function $F : X \times [0,1] \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.*

*The map $F : X \times [0,1] \to Y$ is a **homotopy** of f_0, and the set $\{f_s : s \in [0,1]\}$ is a **homotopy family**.*

Definition 8. *A property is said to be **stable** if, for every function $f_0 : X \to Y$ possessing that property, and for every homotopy $F : X \times [0,1] \to Y$ of f_0, there exists an $\epsilon > 0$ such that for all s, $0 \leq s < \epsilon$, f_s also possesses the property.*

This means: if a function f_0 has a stable property, then for any direction in which we would like to deform f_0 (that is, for any homotopy F) we can always deform it a bit (any amount less than ϵ) without effecting our stable property.

Proposition 9. *For smooth maps from a compact manifold X to any manifold Y, transversality with respect to a given submanifold $Z \subseteq Y$ is a stable property.*

Proof. The transversality of a given map $f_0 : X \to Y$ with respect to $Z \subseteq Y$ is a condition on the span of $D_a f_0$. For any homotopy of f_0, we will use the continuity of the Jacobian to show that all maps in some homotopy neighborhood of f_0 are also transverse to Z. [2]

Let $F : X \times [0,1] \to Y$ be any homotopy of a map $f_0 : X \to Y$, where $f_0 \pitchfork Z$ for some submanifold Z of Y.

Pick $a \in f^{-1}(Z)$. Let k be the codimension of Z in Y. Since $f_0 \pitchfork Z$, we can pick charts on X and Y such that the first $k \times k$ entries of the Jacobian of f_0 form a $k \times k$ submatrix with non-zero determinant (otherwise, our $D_a f_0$ would not span have a sufficient rank to satisfy transversality).

Now, $f_0(a) = F(a, 0)$, and so $D_a f_0 = D_{(a,0)} F$. Since the partial derivatives are all continuous, there is some open interval around $(a, 0) \in X \times [0, 1]$ where this $k \times k$ matrix will always has nonvanishing determinant.

Since X is compact, we can cover $X \times 0$ with a finite number of such open neighborhoods, and then find $\epsilon > 0$ such that $X \times [0, \epsilon)$ is contained in their intersection.

Then, for every $0 \leq t \leq \epsilon$, the Jacobian of $f_t(x) = F(x, t)$ starts with a $k \times k$ submatrix with nonzero determinant. Then $D_a f_t + T_a Y = T_a Z$, and we have found a homology neighborhood of f_0 where $f_t \pitchfork Z$.

□

The preceding proof concerns the stability of $f \pitchfork Z$ under small deformations of f. Transversality is just as stable under small deformations of Z. Similarly, for M and N submanifolds of Y, $M \pitchfork N$ is stable under small deformations of either manifold. To see why this is the case, notice that $M \pitchfork N$ is equivalent to $i \pitchfork N$, where $i : M \to Y$ is the inclusion map. Slight deformations of M are equivalent to slight deformations of the map i, which, as we have proved above, do not effect its transversality with N.

Recall that our original definition of transversality considered not just how two submanifolds intersect, if they intersect, but it also depended on the choice of manifold in which they are embedded.

Consider the example of a pair of intersecting lines in the xy-plane (i.e. two one dimensional submanifolds of a 2-dimensional manifold). If they are transverse, then we can always deform them a bit in any direction without making them tangent or undoing the intersection. This is example of the stability of transversality.

Let us now consider our two lines as submanifolds of $\mathbb{R}^3$. No matter how nice their intersection is, they will never be transverse, since their tangent spaces can have at most a two-dimensional span. However, now we can deform them by an arbitrary small amount – say, lifting one of them out of the xy-plane by some arbitrarily small ϵ – and suddenly they do not intersect, and they have become transverse.

This is an example of an even stronger property than stability. Transversality is also *generic*, meaning that any non-transversal intersection can be deformed by some arbitrarily small amount into a transversal one. To prove that transversality is generic, we will require a bit more mathematical baggage, beginning with Sard's Theorem.

2.4 Sard's Theorem

Let us return to our smooth map $f : X \to Y$. Recall that the points $y \in Y$ such that $D_a f$ is surjective $\forall a \in f^{-1}$ are called **regular values**.

Clearly, regular values are very important, as we can only apply the Preimage Theorem when we have regular values, and the whole notion of transversality is clearly wrapped up with the search for regular values. Given a smooth map f, we might be concerned that regular values are few and far between. Happily, Sard's Theorem tells us that the opposite is true. It is **critical values** (the points $y \in Y$ which are not regular values) which occur very rarely.

Before stating the theorem, let us clarify the concept of **measure zero** for a set in our manifold.

Recall:

Definition 10. *In $\mathbb{R}^n$, a set is said to have* ***measure zero*** *if we can cover it with a countable union of rectangles of arbitrarily small size.*

Now, given a manifold M with a (countable) atlas (U_α, ϕ_α), we can consider a subset L in M to be of measure zero if, $\forall \alpha$, $\phi_\alpha(L)$ has measure zero in $\mathbf{R}^m$.

Theorem 11. *Sard's Theorem.*
Let $f : X \to Y$ be any smooth map of manifolds. Then the set of critical values of f has measure zero.

The proof of Sard's Theorem contains a lot of measure theory, and is well beyond the scope of this paper. We refer the reader to Appendix 1 of Guilleman and Pollack [3].

A property which fails on a set of measure zero can equivalently be said to be true almost everywhere, or true for almost every value. Thus an equivalent statement of Sard's Theorem reads:

Almost every $y \in Y$ is a regular value of f.

2.5 Thom's Transversality Theorem.

In **2.3** we proved that, if we have a function that is transverse to a manifold, we can nudge either one a bit and the result will still be transverse. Thus we have already seen that transversality is quite a resilient condition. In this section we will prove something even stronger: Whenever a smooth function is *not* transverse to a submanifold Z, we can always nudge it slightly (arbitrarily slightly!) into a function which *is* transverse to Z.

Thus the property of transversality is **generic**, and almost all maps are transverse (to Z). We can now give an intuitive justification for having originally defined transversality of two submanifolds in such a way that non-intersecting manifolds are automatically transverse. Returning to our previous example of two intersecting lines in $\mathbb{R}^3$, we want to insure that we can nudge one of them (arbitrary) slightly, with the result that they are transverse. We do this by nudging them (a tiny little bit) in such a way that they don't intersect. If non-intersecting lines *weren't* transverse in $\mathbb{R}^3$, then transversality would not be generic!

Before we get to the statement of the theorem, let X and Y be two manifolds, where (only) X has a boundary.

Recall that, for a smooth function $f : X \to Y$ we can denote by ∂f the restriction of f to ∂X. Then, if Z is a submanifold of Y and we have both $f \pitchfork Z$ and $\partial f \pitchfork Z$, we can apply the generalized Preimage Theorem, and we see that $f^{-1}(Z)$ is a manifold with boundary. Moverover,

$$\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X$$

For the statement of Thom's Transversality Theorem, we will consider a family of smooth maps, $f_s : X \to Y$, indexed over some boundaryless manifold S. We insure that the family $\{f_s\}$ vary smoothly by setting $F : X \times S \to Y$, where $F(x, s) = f_s(x)$, and requiring that F is a smooth function.

Theorem 12 (Thom's Transversality Theorem)**.** *Suppose that $F : X \times S \to Y$ is a smooth map between manifolds, where only X has boundary, and let Z be any boundaryless submanifold of Y. If both F and ∂F are transverse to Z, then for almost every $s \in S$, both f_s and ∂f_s are transverse to Z.*

Proof. I follow the proof laid out in Guilleman and Pollack. [3]

Let $W = F^{-1}(Z)$, and let $\pi : X \times S \to S$ be the natural projection map.

We shall show that $f_s \pitchfork Z$ whenever $s \in S$ is a regular value of the restriction $\pi|_W$, and $\partial f_s \pitchfork Z$ whenever s is a regular value of the restriction $\partial\pi|_W$. Since, by Sard's Theorem, almost every $s \in S$ is a regular value for both maps, the theorem follows.

Since $F \pitchfork Y$, W is a submanifold of $X \times S$ and has boundary

$$\partial W = W \cap \partial(X \times S).$$

Pick $s \in S$ which is a regular value of $\pi|_W$. In order to show that $f_s \pitchfork Z$, pick $x \in f_s^{-1}(Z)$ and let $f_s(x) = z \in Z$.

Since $F(x, s) = z$ and we have by assumption that $F \pitchfork Z$, we know that

$$DF_{(x,s)}(X \times S) + T_z Z = T_z Y$$

that is, given any vector $a \in T_z Y$, there exists a vector $b \in T_{(x,s)}(X \times S)$ such that

$$DF_{(x,s)}(b) - a \in T_z Z.$$

In order to show that $f_s \pitchfork Z$, we need to find a vector $v \in T_x X$ such that $D(f_s)_x(v) - a \in T_z Z$ for our given $a \in T_z Y$.

Since

$$T_{(x,s)} X \times S = T_x X \times T_s S$$

and therefore $b = (w, e)$ for some vectors $w \in T_x X$ and $e \in T_s S$.

Now, if we are very lucky and $e = 0$, then we are done: Since the restriction of F to $X \times s$ is f_s, we have

$$DF_{(x,s)}(w, 0) = D(f_s)_x(w)$$

so $w = v$ is the vector we were looking for, which satisfies $D(f_s)_x(w) - a \in T_z Z$.

If $e \neq 0$, we can use the projection map $\pi : X \times S \to S$ to kill it off. Recall that, by assumption, our point s is a regular value of the $\pi|_W$.

Since

$$D\pi_{(x,s)} : T_x X \times T_s S \to T_s S$$

is just the projection onto the second factor, the regularity assumption (that $D\pi$ is surjective at every point $\pi^{-1}(s) \cap W$) tells us that there is some vector of the form (u, e) in $T_{(x,s)} W$. But $F : W \to Z$, so $DF_{(x,s)}(u, e) \in T_z Z$. Consequently, we can take $v = w - u$ to be our solution. To see why, observe that

$$\begin{aligned} D(f_s)_x(v) - a &= DF_{(x,s)}\big((w, e) - (u, e)\big) - a \\ &= DF_{(x,s)}(w, e) - DF_{(x,s)}(u, e) - a \\ &= DF_{(x,s)}(b) - \big(DF_{(x,s)}(u, e) - a\big) \end{aligned}$$

is in $T_z Z$, since the last two vectors are both in $T_z Z$. Therefore, $f_s \pitchfork Z$

Precisely the same argument holds to show that $\partial f_s \pitchfork Z$ whenever s is a regular value of $\partial\pi$. It can be seen as a special case of the argument we just proved, for the boundaryless manifold ∂X.

□

To see why Thom's Transversality Theorem proves that transversality is generic, consider the special (but clearly generalizable) case of a smooth function f from a manifold M to a target space $Y = \mathbb{R}^n$. Take S to be an open ball in $\mathbb{R}^n$, and consider the function

$$\begin{array}{rlcl} F: & X \times S & \to & \mathbb{R}^n \\ & (x,s) & \mapsto & f(x)+s \end{array}$$

Fixing x, $F(x,s) = f(x) + s$ is just a constant translation of the open ball S. Therefore, even without letting x vary, we can see that $DF_{(x,s)}$ surjects onto $T_{F(x,s)}\mathbb{R}^n$. Therefore, for any Z submanifold of $\mathbb{R}^n$, $F \pitchfork Z$. By the Transversality Theorem, it follows that for almost every $s \in S$, the map $f_s(x) = f(x) + s$ is transversal to Z. Thus even if our original map f is not transverse to Z, we can nudge it arbitrary little in some direction s to achieve a map which *is* transverse to Z.

3 Examples

Here are a few examples of problems that appeared on recent UIC preliminary exams, which feature transversality.

3.1

Let X be a submanifold of $\mathbb{R}^n$. Prove that for almost every $a \in \mathbb{R}^n$, the translate $X + a$ intersects X transversely.

Proof. This is in fact the same application of the Transversality Theorem which we just used to show that transversality is a generic property.

Let

$$\begin{array}{rlcl} F: & X \times \mathbb{R}^n & \to & \mathbb{R}^n \\ & (x,a) & \mapsto & x+a \end{array}$$

Fixing x, we see that DF is surjective onto $\mathbb{R}^n$. Therefore, F is transverse to any submanifold of $\mathbb{R}^n$, in particular to X.

Let $f_a(x) = F(x,a)$. Then $f_a(X) = X + a$, and, by the Transversality Theorem, $f_a \pitchfork X$ for almost every $a \in \mathbb{R}^n$.

□

3.2

For $a \in \mathbf{R}$, let $H_a = \{(x,y,z) \in \mathbf{R}^3 : x^2 + y^2 - z^2 = a\}$.

1. Prove that H_a is a submanifold of $\mathbf{R}^3$ for all $a \neq 0$.
2. For which $a \neq 0$ and $r > 0$ is H_a transverse to the sphere of radius r centered at $(0,0,0)$?

Proof. 1. Consider the map

$$\begin{array}{rlcl} F: & \mathbb{R}^3 & \to & \mathbb{R} \\ & (x,y,z) & \mapsto & x^2+y^2-z^2 \end{array}$$

Then $H_a = F^{-1}(a)$. Let $a \neq 0$. In order to apply the Preimage Theorem, we need to check that DF_p is surjective, for every $p = (p_1, p_2, p_3) \in F^{-1}(0)$. DF_p is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}$, given by

$$DF_p(x, y, z) = (2p_1, 2p_2, -2p_3) \circ (x, y, z) = 2p_1x + 2p_2y - 2p_3z$$

and we are mapping to $\mathbb{R}$, the derivative will be surjective whenever it is not identically zero. Therefore it is enough to find a single vector (x, y, z) such that $DF_p(x, y, z) \neq 0$. Let us take the vector $p = (p_1, p_2, p_3)$. Then,

$$DF_p(p_1, p_2, p_3) = 2(p_1^2 + p_2^2 - p_3^2) = 2a \neq 0$$

since $p \in H_a$. Therefore DF_p is surjective $\forall p \in F^{-1}(a)$, and $H_a = F^{-1}(a)$ is a manifold.

2. Since H_a and the sphere both have tangent spaces of dimension 2, they will be transverse as long as they do not intersect tangentially. If $a < 0$, then H_a is a one sheet hyperboloid, and it is not tangent to the sphere of any radius centered at $(0, 0, 0)$. Therefore, H_a intersects every such sphere tangentially.
   If $a > 0$, then H_a is a two sheet hyperboloid, which is not defined for $|z| < a$. Therefore, whenever the sphere has radius $r < a$, the sphere and H_a do not intersect and they are (trivially) transverse. Whenever $r > a$, their intersection forms two circles (two one dimensional manifolds!) and they are not tangent to eachother, so they are transverse. H_a and the sphere only intersect tangantially when $a = r$.

□

3.3

Suppose that S_1 and S_2 are smooth surfaces in $\mathbb{R}^3$ that intersect at a point p and do not have the same tangent plane at that point. Show that p is not an isolated point of $S_1 \cap S_2$

Proof. We will apply the stability of transversality to a neighborhood of p, and show that $S_1 \cap S_2$ is (in that neighborhood) a manifold of dimension at least 1.

Since S_1 and S_2 are smooth surfaces, their tangent spaces are each two dimensional. Since they do not have the same tangent plane at p, by counting dimensions we have

$$T_pS_1 + T_pS_2 = T_p\mathbb{R}^3$$

Since transversality is stable, we know that there is some open neighborhood B^3 around p where $(S_1 \cup B^3) \pitchfork (S_2 \cup B^3)$.

Let's say $Z = (S_1 \cup S_2) \cup B^3$. Then transversality of S_1 and S_2 restricted to B^3 tell us that Z is a manifold, of dimension at least $3 - 2 = 1$. Therefore, p is a point in a (at least) 1 dimensional manifold contained in $S_1 \cap S_2$, and it is not isolated.

□

4 Transversality in Economics

We have seen that transverality is a very nice condition indeed: once two objects are transverse, they tend to remain transverse under small perturbations, and better yet, any time two objects that are not transverse can easily be perturbed into being transverse.

The stability and genericity of transversality make it a very powerful condition, and give rise to a number of applications, in many branches of science which might not initially seem related to differentiable manifolds. If a data set can be represented as a manifold, which is transverse to some condition that we care about, then we know that any (small) perturbations of the data set will not effect its relation to our important condition.

The economists Kalman and Lin apply transversality to prove local uniqueness of equilibrium states for "almost all" economies [5]. They consider a family $\Sigma = \{E\}$ of (models of) economies, and they associate a map ψ_E to each economy. They define a *regular economy* as one that is transversal with respect to the origin, i.e. $\psi_E \pitchfork 0$. Once the result is proven for regular economies, the genericity and stability of transversality imply that any economy can be approximated by a regular economy, and any regular economy is still regular under small perturbations of economic data in the model. Thus the result is proven for "almost all" economies.

References

[1] Francis, John, *Math 465, Lecture 3: Thom's Theorem*, http://math.northwestern.edu/ jnkf/classes/mflds/3thom.pdf, 2010.

[2] Gualtieri, Marco, *Geometry and Topology Notes*, http://www.math.toronto.edu/mgualt/MAT1300/, 2009.

[3] Guilleman, Victor and Pollack, Allan, *Differential Topology*, Prentice-Hall, 1974.

[4] Hitchin, Nigel, *Differentiable Manifolds* , www.maths.ox.ac.uk/ hitchin/, 2012.

[5] Kalman, Peter and Lin, Kuan-Pin, Applications of Thom's Transversality Theory and Brouwer Degree Theory to Economics, Journal of Mathematical Analysis and Applications, 1979

[6] Mueger, Michael, *An Introduction to Differential Topology, de Rham Theory and Morse Theory*, http://www.math.ru.nl/ mueger/di notes.pdf , 2005.

[7] Thom, Rene, *Quelques proprietes des varietes differentiables*, CM Helvetici, 1954.