ABSTRACT DIFFERENTIAL GEOMETRY VIA SHEAF THEORY

ARDA H. DEMIRHAN

ABSTRACT. We examine the conditions for uniqueness of differentials in the abstract setting of differential geometry. Then we'll come up with the same unique differential associated to a smooth map as in classical set up. Finally, we'll see that the category of smooth manifolds forms a full subcategory of differential triads which will have some applications.

Acknowledgements. The author is thankful to Prof. Laura Schaposnik for suggesting this subject, and for many helpful discussions.

1. Introduction

Abstract differential geometry is a form of differential geometry without the classical calculus notion of smoothness and this is introduced and developed mainly by Anastasios Mallios et al. from 1998. Throughout its development, the theory has been quite general and in a sense axiomatic. Basic tools in that perspective are sheaves of modules with respect to appropriate sheaf of $\mathbb{C}$-algebras so to speak, sheaf theory plays a great role in the articles of Mallios. For example, ever growing usage of a sheaf-theoretic language together with standard differential geometry leads to "geometrization of physical thought " which lead to recent papers involving "gauge theories."

In the beginning of this survey, we'll introduce the notion of categories. Then we'll define a presheaf and a sheaf and give examples for each of them. We define stalks and we describe the process of sheafification using stalks.

Then we'll give an equivalent definition for a sheaf and see why they are equivalent. The whole purpose is to use that definition of sheaves to define sheaf theroretic tools easily.

After introducing basic sheaf theoretical notions we'll begin examining abstract differential geometry as opposed to classical differential geometry.

In classical differential geometry ($\mathcal{CDG}$) one works with smooth manifolds; on the other hand, in the framework of abstract differential geometry ($\mathcal{ADG}$) that we focus in this survey, we have a sense of "freedom" in such a way that we don't have many constraints and we can extend a great deal of $\mathcal{CDG}$ to spaces which are not smooth. Consequently, this will have useful applications to spaces with singularities and modern physics.

ADG replaces the structure sheaf of smooth functions by a sheaf of abstract algebras together with its differential taking values in a module over those algebras. With that data we'll define *Differential Triads* and then make them into a category, namely $\mathcal{DT}$ by introducing suitable morphisms between them. Then we can treat the category of smooth manifolds $\mathcal{M}an$ as a subcategory sitting in $\mathcal{DT}$.

It turns out that differentials of maps between manifolds coincide in both $\mathcal{CDG}$ and $\mathcal{ADG}$ provided some conditions. This will enable us to see why $\mathcal{CDG}$ is a full subcategory

Wednesday 2^{nd} December, 2015.

of $\mathcal{ADG}$.

In this survey, smooth manifolds are assumed to be second countable and Hausdorff. Further, we'll impose that they are finite dimensional.

2. Preliminaries

2.1. Categories: A Brief Introduction.

A model of axiomatic set theory, in particular ZFC[1], is a commonly preferred way to describe whole mathematics.

On the other hand, over the last few decades category theory as an alternative set theory has been developed. In itself, a category is a relaxed version of a set[2] with some structure preserving morphisms. More precisely, a category is defined as follows:

Definition 1. *A* **category** $\mathcal{C}$ *is a quadruple* $(Ob(\mathcal{C}), Hom(\mathcal{C}), \circ, 1)$ *satisfying the following:*

(1) $Ob(\mathcal{C})$ *is a class of objects.*
(2) $Hom(\mathcal{C})$ *is a class consisting of sets,* $Mor_{\mathcal{C}}(a,b)$*, for any given* $a, b \in Ob(\mathcal{C})$*.*
(3) *For any* $a, b, c \in Ob(\mathcal{C})$*:*
$$\circ : \; Mor_{\mathcal{C}}(a,b) \times Mor_{\mathcal{C}}(b,c) \to Mor_{\mathcal{C}}(a,c), \quad (f,g) \longmapsto g \circ f$$
(4) *There is a morphism* $1_a \in Mor_{\mathcal{C}}(a,a)$ *called the identity morphism.*
(5) **Associativity***: For any* $f \in Mor_{\mathcal{C}}(a,b)$*,* $g \in Mor_{\mathcal{C}}(b,c)$*, and* $h \in Mor_{\mathcal{C}}(c,d)$*;* $h \circ (g \circ f) = (h \circ g) \circ f$*.*
(6) **Identity***: For any morphism with domain a and codomain b,* $1_b \circ f = f = f \circ 1_a$*.*
(7) **Disjointness***: If (a,b)≠(c,d) are pairs of objects, then* $Mor_{\mathcal{C}}(a,b) \cap Mor_{\mathcal{C}}(c,d) = \varnothing$.[3]

Let us give some natural examples of categories:

Example 1.1. Set,Grp,Rng,R-Mod,Top can be given as examples of categories where sets, groups, rings, (left) R-modules and topological spaces form class of objects and functions, group homomorphisms, ring homomorphisms, homeomorphisms form class of morphisms respectively.

As always, we have "sub"categories, here our emphasis is on full subcategories because this notion will appear in the last result of this survey.

Definition 2. *Let* $\mathcal{C}$ *be a category. Then* $\mathcal{D}$ *is a* **subcategory** *of* $\mathcal{C}$ *if* $Ob(\mathcal{D})$ *is a subclass of* $Ob(\mathcal{C})$ *and for any objects* a, b *in* $\mathcal{D}$*,* $Mor_{\mathcal{D}}(a,b) \subseteq Mor_{\mathcal{C}}(a,b)$ *with the same composition and identity elements. Furthermore, if for any two objects in* $\mathcal{D}$ *we have* $Mor_{\mathcal{D}}(a,b) = Mor_{\mathcal{C}}(a,b)$ *then* $\mathcal{D}$ *is called a* **full subcategory.**

---

[1]ZFC is a model of axiomatic set theory constructed by Zermelo and Fraenkel; C stands for Axiom of Choice.

[2]In the sense that we don't have to worry about Russell's Paradox.

[3]For this reason, a function in the category **Set** is defined as a triple $\widetilde{f}$=(f,A,B) where A is the domain and B is the range of f and f is a function in the traditional sense.

2.1.1. **Functors Between Categories**

Naively, a functor between categories is analogous to a function between sets. Formally, it can be defined as follows:

Definition 3. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then a* **functor** $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ *consists of two functions$Ob(\mathcal{C}) \to Ob(\mathcal{D})$, and $Hom(\mathcal{C}) \to Hom(\mathcal{D})$ both denoted by $\mathscr{F}$ so that:*

(1) $\mathscr{F} : Mor_{\mathcal{C}}(a, b) \to Mor_{\mathcal{D}}(\mathscr{F}(a), \mathscr{F}(b))$.
(2) $\mathscr{F}(1_a) = 1_{\mathscr{F}(a)}$, *for each* $a \in Ob(\mathcal{C})$.
(3) $\mathscr{F}(fg) = \mathscr{F}(f)F(g)$, *for all f, g that can be composable.*

By ZFC, any set has a power set and functions between two sets can be restricted to subsets so that these two naturally defines a functor:

Example 3.1. *The Power Set Functor*

$$\begin{aligned} \mathscr{P} : \quad & \mathbf{Set} \to \mathbf{Set}, \\ & S \mapsto \mathscr{P}(S) \\ (f : S \to T) \longmapsto (\mathscr{P}(f) : \quad & \mathscr{P}(S) \to \mathscr{P}(T)) \\ & u \mapsto f(u) \end{aligned}$$

2.1.2. **Natural Transformations**

One wants to consider a category with objects as functors between two given categories, then morphisms of this category must be constructed, these are called natural transformations.

Definition 4. *Let $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathcal{D}$ be functors. A* **natural transformation** *is a class of morphisms τ consisting of τ_a where a runs over $Ob(\mathcal{C})$ and $\tau_a : \mathscr{F}(a) \to \mathscr{G}(a)$ is a morphism in $\mathcal{D}$ such that the following diagram commutes:*

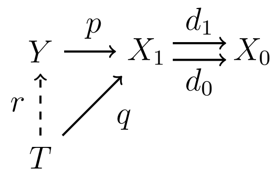
$$\begin{array}{ccc} \mathscr{F}(a) & \xrightarrow{\tau_a} & \mathscr{G}(a) \\ {\scriptstyle \mathscr{F}(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{G}(f)} \\ \mathscr{F}(b) & \xrightarrow[\tau_b]{} & \mathscr{G}(b) \end{array}$$

If for each a, τ_a is an isomorphism[4] *then, we write $\mathscr{F} \cong \mathscr{G}$ so that τ is called as natural isomorphism. Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be* **equivalent** *if there are two functors $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ and $\mathscr{G} : \mathcal{D} \to \mathcal{C}$ so that $\mathscr{F}\mathscr{G} \cong 1_{\mathcal{D}}$ and $\mathscr{G}\mathscr{F} \cong 1_{\mathcal{C}}$ where $1_{\mathcal{C}}$ and $1_{\mathcal{D}}$ are identity functors.*

Before starting the next section, let's define one more category theoretic notion which will be usual to discuss sheaves:

Definition 5. *Let $d_1, d_0 : X_1 \to X_0$ be in a category $\mathcal{C}$. Then a pair $(Y, p : Y \to X_1)$ is said to be an* **equalizer** *if $d_1 \circ p = d_0 \circ p$ and it is universal with respect to this property, that is, for any $q : T \to X_1$ with $d_1 \circ q = d_0 \circ q$, there is a unique morphism $r : T \to Y$ such that the following diagram*

[4]Isomorphism means invertible morphism.



commutes.

2.2. **Sheaves: Definition and Examples.**

In this section, we'll introduce the notion of sheaves which are basically "nice" presheaves which we will define now.

Definition 6. *Let $\mathcal{C}$ be a category, X be a topological space, and $\mathbf{Op}(\mathbf{X})$ denote the category of open subsets of X where the only morphisms are inclusions of open sets. Then a* **presheaf** *$\mathscr{F}$ is a contravariant functor*

$$\begin{aligned} \mathscr{F} : \mathbf{Op}(\mathbf{X}) &\to \mathcal{C} \\ U &\mapsto \mathscr{F}(U). \end{aligned}$$

Unraveling the definition of a contravariant functor a sheaf is determined by the following data:

1. *For any open set U in X, there is an open set $\mathscr{F}(U)$ associated to U.*
2. *If both $V \subseteq U$ are open sets, then there is an associated $\mathcal{C}$-morphism $\mathscr{F}_{V\subseteq U} : \mathscr{F}(U) \to \mathscr{F}(V)$.*
3. *For an open set U, $\mathscr{F}_{U\subseteq U} : \mathscr{F}(U) \to \mathscr{F}(U)$ is the identity map on $\mathscr{F}(U)$.*
4. *If we have three open sets $W \subseteq V \subseteq U$ then $\mathscr{F}_{W\subseteq V} \circ \mathscr{F}_{V\subseteq U} = \mathscr{F}_{W\subseteq U}$*

Remark 7. *It is convenient to denote $\mathscr{F}(U)$ as the sections s of $\mathscr{F}$ over U and we refer to $\mathscr{F}_{V\subseteq U}$ as restriction maps. Then for $s \in \mathscr{F}(U)$*[5] *we write $s\big|_V$ instead of $\mathscr{F}_{V\subseteq U}(s)$.*

Given two sheaves $\mathscr{F}, \mathscr{G} : \mathbf{Op}(X) \to \mathcal{C}$, a morphism between them is a $\mathcal{C}$-morphism $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ for any open set $U \subseteq X$ making the following diagram commutative:

$$\begin{CD} \mathscr{F}(U) @>{\phi(U)}>> \mathscr{G}(U) \\ @V{\mathscr{F}_{V\subseteq U}}VV @VV{\mathscr{G}_{V\subseteq U}}V \\ \mathscr{F}(V) @>>{\phi(V)}> \mathscr{G}(V) \end{CD}$$

Definition 8. *Let $\mathscr{F} : \mathbf{Op}(X) \to \mathcal{C}$ be a presheaf and U be an open set with an open covering $\{U_i\}_{i\in I}$. Then $\mathscr{F}$ is said to be a* **sheaf** *if it satisfies three properties:*

[5] To be accurate, from now on we assume that $\mathcal{C}$ is a concrete category, that is, a category admitting a faithful functor to the category of sets **Set** so objects in $\mathcal{C}$ are sets with some structure, so picking an element from an object makes sense

(1) *The underlying set of* $\mathscr{F}(\varnothing)$ *is a singleton.*[6]

(2) *Given* $s, t \in \mathscr{F}(U)$, *then* $s = t$ *if* $s\big|_{U_i} = t\big|_{U_i}$ *for any* $i \in I$.

(3) *If* $s_i \in \mathscr{F}(U_i)$ *and* $s_j \in \mathscr{F}(U_j)$ *and they satisfy* $s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j}$ *for any* $i, j \in I$, *then there is an element* $s \in \mathscr{F}(U)$ *with* $s\big|_{U_i} = s_i$ *for any* $i \in I$.

Morphisms between sheaves are precisely morphisms between presheaves.

Remark 9. *Actually,* $s \in \mathscr{F}(U)$ *in* (3) *is unique after assuming* (2).

Remark 10. *We can sum all these statements up by saying that* $\mathscr{F}$ *is a sheaf if and only if*

$$\mathscr{F}(U) \longrightarrow \prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} \mathscr{F}(U_{ij})$$

is an equalizer for any open U with cover $\{U_i\}_i \in I$ Now it's time to see some examples:

Example 10.1. Let X be a topological space. For any open set $U \subseteq X$, let $\mathscr{F}(U)$ be the set of continuous of real valued functions.
Let Y be a smooth manifold. For any open set $V \subseteq Y$, let $\mathscr{G}(V)$ be the set of smooth maps on V.
Let Z be a complex manifold. For any open set $W \subseteq Z$, let $\mathscr{H}$ be the set of holomorphic functions on W.
Then $\mathscr{F}, \mathscr{G}, \mathscr{H}$ are sheaves of sets on X, Y, and Z respectively where the restriction morphisms are in the usual sense.

Example 10.2. Let X be a topological space, $p \in X$ and A be an abelian group. For any open set $U \subseteq X$, let

$$S(U) = \begin{cases} A & \text{if } p \in U, \\ 0 & \text{if } p \notin U. \end{cases}$$

With that construction, morphisms are forced to be either the identity or 0 homomorphisms. S is a sheaf of abelian groups.

Example 10.3. Let $f : X \to Y$ be a continuous map of topological spaces X and Y. Let $Sh(X)$ and $Sh(Y)$ be the category of sheaves on X and Y respectively.
Now let $f_* : Sh(X) \to Sh(Y)$ be a functor taking a sheaf $\mathscr{F}$ to $f_*\mathscr{F}$ given by pushout, that is, we set $f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U))$.
f_* is called the **direct image functor**.

Before moving on, we look at non-examples:

Example 10.4. Let X be a topological space, for any open set $U \subseteq X$ define $\mathscr{F}(U)$ as the set of real valued bounded functions on U and let the restriction maps be as usual. So we obtain a presheaf on X. Then if we have bounded functions s_i on U_i and s_j on U_j with $s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j}$ for any $i, j \in I$ then there is a unique extension map s on U but s may not be bounded. So $\mathscr{F}$ is not a sheaf.

[6]Some authors include this in the definition of a presheaf.

Example 10.5. Let X be a topological space and let G be a abelian group. For any open set $U \subseteq X$ let $\mathscr{G}(U)$ be the group of of constant functions from U to G with componentwise multiplication and usual restriction maps. Then $\mathscr{G}$ is a presheaf. However, it may not be a sheaf.
For example, let's give $X = \{1,2\}$ the discrete topology and let $G = \mathbb{Z}$. Then say $U_i = \{i\}$ for $i = 1,2$ and suppose $s_1 = 0$ and $s_2 = 1$. Then there is no constant function which restricts to both 0 and 1.
We can remedy this situation, we just let $\mathscr{G}^+$ to be the sheaf of locally constant functions then it becomes a sheaf.

The last example gives some motivation on how to modify a presheaf to make it a sheaf, to do this precisely, we need the following notion:

Definition 11. *Let X be a topological space and let $\mathscr{F}$ be a presheaf*[7] *on X. Let $x \in X$. The* **stalk** *of $\mathscr{F}$ at x, is defined to be the inverse limit*

(1)
$$\mathscr{F}_x := \varinjlim_{U \ni x} \mathscr{F}(U)$$

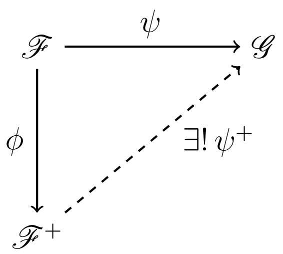
Unraveling the definition of a direct limit, this is equivalent to say that a representative of an element of $\mathscr{F}_x$ is a pair (U,s) where U is an open set containing x and $s \in \mathscr{F}(U)$, and, we say that two representatives (U,s) and (V,t) give the same element in $\mathscr{F}_x$ if there is an open set $x \in W \subseteq U \cap V$ such that $s\big|_W = t\big|_W$.
Elements of the stalk $\mathscr{F}_x$ are referred as **germs** *of sections of $\mathscr{F}$ at x.*

Now we can describe the process of sheafification:

Definition 12. *Let $\mathscr{F}$ be a presheaf on a topological space X. For any open set $U \in X$, we define $\mathscr{F}^+(U)$ as follows:*
(2)
$$\mathscr{F}^+(U) := \left\{ (t_x)_{x\in U} \in \prod_{x\in U} \mathscr{F}_x \,\middle|\, \exists \text{ cover } \{U_i\}_{i\in I} \text{ of } U \text{ and } s_i \in \mathscr{F}(U_i) \text{ with } t(x) = (s_i)_x \ \forall x \in U_i \right\}$$

where $(s_i)_x$ is the germ in $\mathscr{F}_x$ represented by (U_i, s_i). Then $\mathscr{F}^+$ is a sheaf.

Remark 13. *For any presheaf $\mathscr{F}$, there is a natural morphism of presheaves $\phi : \mathscr{F} \to \mathscr{F}^+$ taking $s \in \mathscr{F}(U)$ to m where $m(x) = t_x$ for any $x \in U$.*
Now the pair $(\mathscr{F}^+, \phi)$ is universal in the sense that for any pair $(\mathscr{G}, \psi)$ consisting of a sheaf and a morphism of presheaves $\psi : \mathscr{F} \to \mathscr{G}$ there is a unique morphism $\psi^+ : \mathscr{F}^+ \to \mathscr{G}$ such that $\psi = \psi^+ \circ \phi$, that is



commutes.

[7]For simplicity say it is a presheaf of sets but the definition applies to the category of abelian groups, rings etc.

Example 13.1. $\mathscr{G}^+$, the sheaf of locally constant functions is the sheafification of $\mathscr{G}$ in Example 10.5.

2.3. **Sheaves: Alternative Point of View.**

The definition that we gave above is quite technical but more common. There is an alternative way to sheaves which is equivalent to our definition which will be useful when we discuss abstract differential geometry.

Definition 14. *Let S and X be topological spaces. Then a* **local homeomorphism** *$f : S \to X$ is a continuous map such that for any $s \in S$ there is an open neighborhood U_s and $f\big|_{U_s} : U_s \to f(U_s)$ is a homeomorphism and $f(U_s)$ is open in X.*

Now, let $f : S \to X$ be a surjective local homeomorphism. For an open set $U \subset X$, set $\mathscr{F}(U)$ to local sections s of f, that is, $s : U \to S$ such that $f \circ s = id_U$. Then $\mathscr{F}$ becomes a sheaf.

Moreover, $f^{-1}(x) = \mathscr{F}_x$ the stalk of $\mathscr{F}$ at x.

Conversely, given a sheaf $\mathscr{F}$ on X, we set S to the union of stalks with an appropriate topology. The sheafification construction that we described above give the description of this topology.

Remark 15. *By the argument above, we see that there is an equivalence categories between categories of sheaves and the category of local homeomorphisms with codomain X. So this enables us to define a sheaf as follows:*

Corollary 16. *A sheaf is a triple $\mathscr{F} = (S, f, X)$ where $f : S \to X$ is a surjective local homeomorphism*

Example 16.1. All sheaves described in Example 10.1, are the sheaf of local sections of the projection $\pi : X \times \mathbb{F} \to X$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ respectively.
With that motivation we can think of the sheaf of local sections of a tangent bundle, cotangent bundle or more generally of a vector bundle.

To introduce a differential triad, we'll use a sheaf of algebras so we define this notion first.

Definition 17. *An* **algebra** *A is a vector space over a field $\mathbb{F}$ equipped with a bilinear product operation $. : A \times A \to A$.*

Example 17.1. $\mathbb{R}, \mathbb{C}, \mathbb{H}, M_n(\mathbb{R}), \mathbb{R}[x_1, ..., x_n]$ are all $\mathbb{R}$-algebras.

We also require that A is commutative, associative, and unital with respect to that operation so that A becomes a commutative ring with 1. Moreover, our base field is $\mathbb{C}$. As a special type of an algebra we introduce the following example.

Definition 18. *A* **Leibniz algebra** *is a $\mathbb{F}$-module A, admitting a bracket operation, which is $\mathbb{F}$-bilinear and satisfies the Leibniz formula:*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y] \tag{3}$$

A classical example of a Leibniz algebra is a Lie algebra because it satisfies antisymmetry and so Jacobi identity turns into Leibniz identity. Let's see some other examples:

Example 18.1. Let (L, d) be a differential Lie algebra with the Lie bracket $[,]$. Then L is a Leibniz algebra with the bracket operation $[x, y]_d := [x, dy]$. The new bracket on L is called the derived bracket.

3. Essential Notions in Abstract Differential Geometry

Definition 19. *Let X be a topological space. A* **differential triad** *over X is a triplet $\delta = (\mathcal{A}, \partial, \Omega)$ where $\mathcal{A}$ is a sheaf of algebras over X, Ω is an $\mathcal{A}$-module and $\partial : \mathcal{A} \to \Omega$ is a Leibniz morphism.*
Untwisting the definition we have the following data.
$\partial : \mathcal{A} \to \Omega$ is a sheaf morphism such that

(1) *∂ is a $\mathbb{C}$-linear morphism relative to the associated $\mathbb{C}$-vector sheaves $\mathcal{A}, \Omega$ on X,*

(2) *∂ satisfies the Leibniz condition*

$$\partial(x.y) = x.\partial(y) + y.\partial(x), \tag{4}$$

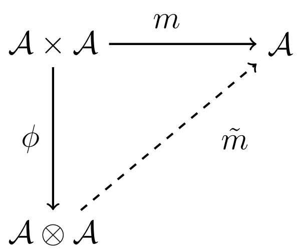
for any $x, y \in \mathcal{A} \times_X \mathcal{A}$; in other words, for any local sections $x, y \in \mathcal{A}(U)$ for any $U \subseteq X$ open.

We give the canonical example first.

Example 19.1. Let X be a smooth manifold. Let $\mathcal{A}$ be the structure sheaf of germs of $\mathbb{C}$-valued function X, let $\Omega = \Omega^1_X$ be the sheaf of germs of its smooth $\mathbb{C}$-value 1-forms on X. So by definition we have, $\Omega^1_X := S(\Gamma(\mathcal{T}^*_{\mathbb{C}(X)}))$; that is, it precisely contains the smooth sections of the complexification of the cotangent bundle. Finally, we set $\partial : \mathcal{A} \to \Omega$ to the sheaf morphism $d_X : \mathscr{C}^\infty_X \to \Omega^1_X$. Precisely, for any $\mathbb{C}$-valued local $\mathcal{C}^\infty$ function on X, d_X sends it to its differential. In a compact form, this means d_X is the sheafification of the usual differential.

Sheafification of Kahler's differential gives rise to another such object.

Example 19.2. Let $\mathcal{A}$ be an algebra. By the universal property of tensor product, there is a unique linear making m, making the below diagram commute



since the multiplication operation $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is bilinear. So for any $(a, b) \in \mathcal{A} \times \mathcal{A}$,

$$\tilde{m}(a \otimes b) = m(ab) =: ab \tag{5}$$

Then,

$$\delta_{\mathcal{A}} : \mathcal{A} \to \ker \tilde{m}/(\ker \tilde{m})^2 \tag{6}$$

$$a \mapsto [(a \otimes 1 - 1 \otimes a)] \tag{7}$$

is a derivation. Then $(\mathcal{A}, \delta_{\mathcal{A}}, \ker \tilde{m}/(\ker \tilde{m})^2)$ is a differential triad, and it's called **Kahler's Differential**.

To obtain the category $\mathcal{DT}$, we need to see how objects are related, in other words, we should define morphisms in that category.

Definition 20. *Let X, Y be topological spaces, let $\delta_X = (\mathcal{A}_X, \partial_X, \Omega_X)$ and $\delta_Y = (\mathcal{A}_Y, \partial_Y, \Omega_Y)$ be differential triads over X and Y respectively. Now a morphism $\tilde{f}$ between them is a triple $(f, f_{\mathcal{A}}, f_{\Omega})$ subject to the following conditions:*

(1) *$f : X \to Y$ is continuous.*

(2) *$f_{\mathcal{A}} : A_Y \to f_*(\mathcal{A}_X)$ is a unit preserving morphism of sheaves of algebras.*

(3) *$f_{\Omega} : \Omega_Y \to f_*(\Omega_X)$ is an $f_{\mathcal{A}}$-morphism.*

(4) *The following diagram*

$$\begin{array}{ccc} \mathcal{A}_Y & \xrightarrow{f_{\mathcal{A}}} & f_*(\mathcal{A}(X)) \\ \partial_Y \big\downarrow & & \big\downarrow f_*(\partial_X) \\ \Omega_Y & \xrightarrow[f_{\Omega}]{} & f_*(\Omega_X) \end{array}$$

is commutative.

Remark 21. *As we promised in the introduction, we can extend the notion of differentiability as follows: we say $f : X \to Y$ is* **differentiable**, *if it can be extended into a morphism $\tilde{f} = (f, f_{\mathcal{A}}, f_{\Omega})$ and then we refer to f_{Ω} as a differential of f.*

4. Main Results

Now with our abstract perspective of the differentiability, we need to deal with two issues. For a pathological sheaf of algebras, there is no guarantee that there is a map extending a map which is differentiable in the classical sense. Moreover, even if such an extension exists it may not be unique.

To remedy the first problem we argue as follows: Let X and Y be topological spaces provided by the differential triads δ_X and δ_Y. Let $y \in Y$ and consider the constant map $\overline{y} : X \to Y$ sending any x to y. Then we obtain:

$$\overline{y}_*(\mathcal{A}_X)(U) = \mathcal{A}(\overline{y}^{-1}(U)) = \mathcal{A}_X(X) \tag{8}$$

for any $y \in U$ where U is any open subset of Y. If $y \notin U$, then $\overline{y}_*(\mathcal{A}_X)(U) = \mathcal{A}(\overline{y}^{-1}(U)) = \varnothing$.

Therefore, if every singleton is closed in Y, that is, if Y satisfies the seperation axiom $\mathcal{T}1$ then $\overline{y}_*(\mathcal{A}_X)$ is a sheaf over $\{y\}$.

Hence our question of the differentiability of the constant map $y : X \to Y$ boils down to the question of the existence of a unit preserving algebra morphism

$$y_{\mathcal{A}} : \mathcal{A}_{Y,y} \to \mathcal{A}_X X \tag{9}$$

and, likewise of a $y_{\mathcal{A}}$-morphism

$$y_\Omega : \Omega_{Y,c} \to \Omega_X(X), \tag{10}$$

making the corresponding diagram which appears in the definition of morphisms in $\mathcal{DT}$

$$\begin{array}{ccc} \mathcal{A}_Y & \xrightarrow{\;y_{\mathcal{A}}\;} & y_*(\mathcal{A}(X)) \\ \partial_Y \downarrow & & \downarrow y_*(\partial_X) \\ \Omega_Y & \xrightarrow[\;y_\Omega\;]{} & y_*(\Omega_X) \end{array}$$

commutative.
Now we have the following:

Theorem 22. *Let $\delta_X = (\mathcal{A}_X, \partial_X, \Omega_X)$ and $\delta_Y = (\mathcal{A}_Y, \partial_Y, \Omega_Y)$ be differential triads over X and Y respectively. Suppose that $\mathcal{A}_Y$ is a functional algebra sheaf over Y, then any constant map $y : X \to Y$ is differentiable.*

Proof. Any complex number is a global section of $\mathcal{A}_X$ because $\mathcal{A}_X$ contains the sheaf $X \times \mathbb{C}$. (Otherwise, $\mathcal{A}_X$ does not contain a unit.)
Now let $x \in \mathcal{A}_{Y,y}$, then x is a germ of an element x' so that $x' \in \mathcal{A}_Y(U) \subseteq \mathcal{C}(U, \mathbb{C})$ where U is an open neighborhood around y.

Now let

$$y_{\mathcal{A}} : \mathcal{A}_{Y,y} \to \mathcal{A}_X(X) \qquad\qquad x \mapsto x'(y) \tag{11}$$

preserves unit as an algebra morphism.
Moreover, if $y_{\mathcal{A}}(a) \in \mathbb{C}$, then $\partial_X \circ y_{\mathcal{A}} = 0$. Thus the 0 morphism makes the corresponding diagram commutative. □

Generalizing this theorem we get the following.

Theorem 23. *If $f, f_{\mathcal{A}}, f_\Omega$ is a morphism in $\mathcal{DT}$, then f_Ω is uniquely determined by $f_{\mathcal{A}}$ on the image $Im\partial_Y$ of ∂_Y.*
Conversely, if ∂_X vanishes only on the constant subsheaf $X \times \mathbb{C} \subseteq \mathcal{A}_X$, then f_Ω uniquely determines $f_{\mathcal{A}}$.

Proof. If $(f, f_{\mathcal{A}}, f_\Omega)$ and $(f, f'_{\mathcal{A}}, f_\Omega)$ are morphisms in $\mathcal{DT}$ then $f_*(\partial_X)(f_{\mathcal{A}}(x) - f'_{\mathcal{A}}(x)) = 0$. Hence, $f_{\mathcal{A}}(x) - f'_{\mathcal{A}}(x) = y$ for any $x \in \mathcal{A}_Y$. By linearity, it is identically zero. □

Remark 24. *Typically, this is a translation argument applied to the previous theorem.*

4.1. **Final Result.**

It turns out that if X and Y are smooth manifolds provided with their smooth differential triads δ_X^∞ and δ_Y^∞ then letting $\tilde{f} = (f, f_{\mathcal{A}}, f_\Omega)$ be a morphism in the category of differential triads, we obtain that f is smooth and $\tilde{f} = F(f)$ where

$$F : \mathscr{M}an \to \mathcal{DT} \tag{12}$$

is the functor on the category of smooth manifolds $\mathscr{M}an$ sending a smooth manifold X to $F(X) := \delta_X^\infty = (\mathcal{C}_X^\infty, d_X, \Omega_X^1)$ (see Example 19.1) and sending smooth maps f to

$F(f) = (f, f_{\mathcal{A}}, f_{\Omega})$ and F is an embedding.

By the remark above, $\mathscr{M}$an becomes a full subcategory of $\mathcal{DT}$.

References

2145 N Humboldt Blvd, Chicago-IL 60647
E-mail address: ademir3@uic.edu