Introduction to Immersion, Embedding, and the Whitney Embedding Theorems

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Abstract

We give an overview of immersion in order to present the idea of embedding, then discuss Whitney's work on his weak immersion and strong embedding theorems as the main theorems of the talk, drawing briefly on notions of transversality as presented by the previous speaker, rigorously proving a weaker form of Whitney's embedding theorem, and sketching out a result of a linear bound for the minimum dimension of the real space in which a manifold of a given dimension can be embedded. We touch briefly on knot-theoretic applications of embeddings, paying particular attention to alternative arguments for the isotopy of all embeddings of S^1 into $\mathbb{R}^4$, before finally addressing a recent preliminary examination problem involving embeddings, surveying recent work in the field, and touching on the importance of embedding to two interesting open problems.

Introduction

In this survey, we will be discussing the concept of immersion as a lead-in to that of embedding, and then finally prove the Whitney embedding theorem for compact manifolds. The Whitney embedding theorem says that any smooth, differentiable real manifold with dimension m can be embedded in $\mathbb{R}^n$. The concepts of immersion and embedding, while natural, were only recently fully formalized, with Whitney's work finishing as late as 1944. Nevertheless, the concept of an embedding in particular formalizes the intuitive notion of an inclusion map from one smoothly differentiable manifold into another.

For simplicity's sake, domains will generally be compact, smoothly differentiable real manifolds, and the codomains of all maps will be some $\mathbb{R}^n$, since we are concerned primarily with classical embedding theory, rather than hyperbolic, complex, or Riemannian embeddings, all of which are mostly beyond the scope of this survey.

Immersion

Let M, N be differentiable manifolds. Then an immersion is a differentiable function $f : M \to N$ whose derivative is everywhere injective. That is, for each point in the domain, there exists some neighborhood of that point in the domain for which locally the image of the neighborhood looks locally like an inclusion into the codomain. Alternatively, an immersion is a map that is injective on the tangent spaces, that is, $D_p f : T_p M \hookrightarrow T_{f(p)} N$ for all points $p \in M$. This means that the image of the manifold under the map can have transverse intersections, since at any such intersection, the tangent spaces of the intersecting parts of the image of the manifold will be distinct, by the definition of transversality. Two instances of immersions can be found below.

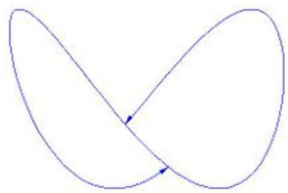

Figure 1: An immersion of an open interval into $\mathbb{R}^2$.

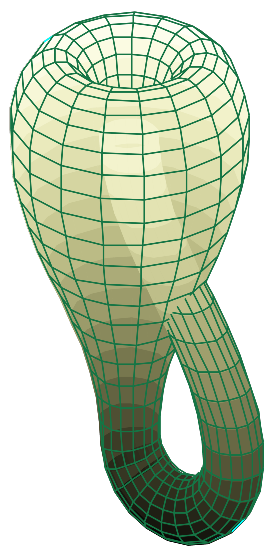

Figure 2: Another immersion, this time of the Klein bottle into $\mathbb{R}^3$.

Topological Embedding

We move for a moment to topology, which has more relaxed standards on spaces than differential geometry does. Let M, N be topological spaces. Then $f : M \to N$ is a topological embedding if f is an injective continuous map which yields a homeomorphism between M and $f(M)$, where $f(M)$ inherits the subspace topology as possessed by N. If we have such an f, then we are able to treat M as a subspace of N, in some sense. What's more, if f is strictly open or closed, then it must be a topological embedding. This is however merely a sufficient condition, as it is possible for $f(M)$ not to be strictly open or closed as a subspace of N, in which case f can't possibly be an open or closed homeomorphism. Below is one possible embedding of S^1 into $\mathbb{R}^2$, a hexagon. We note that we don't need for the map to be smooth or differentiable, or have distinct tangent spaces, merely that the map be homeomorphically injective.

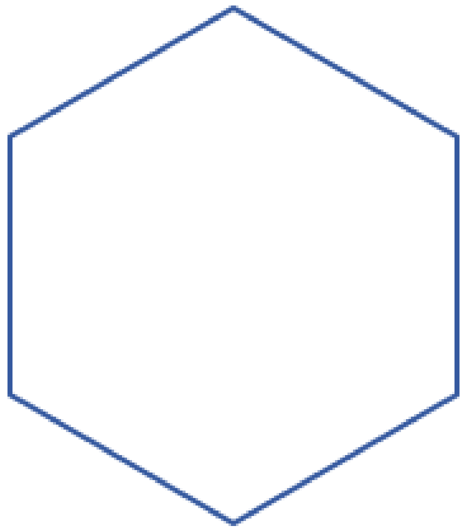

Figure 3: One possible embedding of S^1 into $\mathbb{R}^2$.

Embedding

An immersion on a compact manifold which is also injective is called an embedding. That is, we require that $f : M \hookrightarrow N$ in addition to $D_p f : T_p M \hookrightarrow T_{f(p)} N$ for all points $p \in M$. Essentially, an embedding is an immersion where we no longer allow self-intersections. Thus the image of an embedding is differentiably homeomorphic to its domain, and in particular must be a submanifold, since the diffeomorphic bijection of the map restricted to its own image also defines an atlas on the image when composed with the atlas of the original manifold.

This is a particularly powerful statement about the relationship between two manifolds: an embedding between two manifolds immediately induces an isomorphism between the domain and the image, which tells us a lot about the structure of the codomain. In particular, we know that any immersion which is also a topological embedding is an embedding in the differential geometric sense. Below is one possible embedding of S^2 into $\mathbb{R}^3$; note that since we are dealing merely with differential geometry and nothing stricter that merely being smooth and having no self-intersections suffices.

Isometric Embeddings

We now consider a slightly stricter notion yet than that of the simple differential-geometric embedding: that of the isometric embedding, which belongs properly to Riemannian geometry. Let

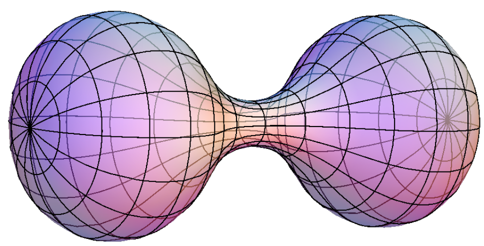

Figure 4: One possible embedding of S^2 into $\mathbb{R}^3$.

$(M,g),(N,h)$ be Riemannian manifolds, that is, manifolds equipped with an inner product defined on the tangent space varying smoothly over the points of the manifold. Then an isometric embedding is a smooth embedding $f : M \to N$ which preserves the metric of M on $f(M) \subset N$, that is, $g = f \circ h$, so that g is equal to the pullback of h by f. Stated more explicitly, for every pair of tangent vectors $v, w \in T_x(M)$, we will have $g(v,w) = h(df(v), df(w))$. In effect we now add the additional requirement that lengths and angles be preserved under the embedding, where naturally we must have a domain and codomain where lengths and angles are meaningful. Most of these embeddings can be viewed as rigid transformations. We can, of course, define analogously the concept of an isometric immersion as paralleling the definition of an immersion.

The Whitney Embedding Theorems

In order to prove the main theorem, we must first do some groundwork.

The Whitney Embedding Theorem: Let M be a manifold. Then there exists some integer N such that $f : M \to \mathbb{R}^N$ is an embedding.

Let U_i, ϕ_i be an open cover of M with associated maps, with finite subcover V_i of cardinality k. Since the V_i have images that are open balls, we can easily embed each one into some $\mathbb{R}^N$, where N is the same across each element of the subcover. Finally, as presented in the text, we can use bump functions f_i subordinate to the cover and define maps:

$$\tilde{\phi}_i(x) = \begin{cases} \phi_i(x) f_i(x), & x \in V_i \\ 0, & x \notin V_i. \end{cases} \tag{1}$$

This will be a set of maps from M to $\mathbb{R}^{dim\ M}$, since the $f_i(x)$ are scalar, and the $\phi_i(x)$ are maps from open balls in M to open balls in $\mathbb{R}^{dim\ M}$. Now we can define a map pasting these all together, defined as follows:

$$\Phi(x) : M \to \mathbb{R}^{k(dim\ M+1)}, x \mapsto (\tilde{\phi}_1(x), \tilde{\phi}_2(x), ..., \tilde{\phi}_k(x), f_1(x), f_2(x), ..., f_k(x)). \tag{2}$$

We can see that this is injective. Define $F_i(x) = (f_1(x), f_2(x), ..., f_k(x)$, and let $x \neq x'$, $\Phi(x) = \Phi(x')$. Then since the $f_i(x)$ are not everywhere 0, and $\Phi(x) = \Phi(x')$, we must also have $F_i(x) = F_i(x')$ and thus for some i, $f_i(x) = f_i(x') \neq 0$, so that $x, x' \in V_i$. But then $\phi_i(x) = \phi_i(x')$ for all i, so that because the ϕ_i are embeddings, $x = x'$, a contradiction. Thus Φ is injective.

It remains to show that $D\Phi$ is also injective. Clearly, we can calculate using the product rule that

$$D\Phi(x) = Df_1(x)\phi_1(x) + D\phi_1(x)f_1(x), ..., Df_k(x)\phi_k(x) + D\phi_k(x)f_k(x), Df_1(x), ..., Df_k(x), \quad (3)$$

which takes points $v \in T_xM$ to vectors in $(\mathbb{R}^{dim\ M})^{\oplus k}) \times \mathbb{R}^{\oplus k}$. But this vector can't be zero, since notevery $f_i(x)$ is 0, and the ϕ_i are embeddings, so that $D\Phi$ is injective. This means that Φ is at least an immersion, and since M is compact, and since Φ is a bijection when restricted to its image, and since any closed $K \subset M$ is compact, $\Phi(K)$ must also be compact and thus closed. This completes the proof.

The Weak Whitney Embedding Theorem: Let M be a manifold of dimension m. Then there exists an embedding $f : M \to \mathbb{R}^{2m+1}$.

We start with an embedding $\Phi : M \to \mathbb{R}^N$ as given by the previous lemma; assume that $N > 2m + 1$, for $m = dim\ M$. Then we can show that for all such N that by projecting to a hyperplane, we can get an embedding to $\mathbb{R}^{N-1}$. A single vector $v \in S^{N-1}$ defines a hyperplane in $\mathbb{R}^N$ as the orthogonal vector; define

$$P_v : \mathbb{R}^N \to \mathbb{R}^{N-1}$$

be the obvious orthogonal projection to the hyperplane. Then it suffices to show that the set of all v for which $\Phi_v = P_v \circ \Phi$ fails to be an embedding is a set of measure 0, so that it must be possible to choose a v for which Φ_v is, in fact, an embedding.

By the definition of an embedding, Φ_v fails to be an embedding if it fails to be injective or $D\Phi_v$ fails to be injective. We consider these two cases separately. First, let $g : (M \times M)\backslash\Delta_M \to S^{N-1}$ be given by

$$g(x_1, x_2) = \frac{\Phi(x_2) - \Phi(x_1)}{||\Phi(x_2) - \Phi(x_1)||}, \quad (4)$$

where Δ_M is the diagonal. Since g is the map taking pairs of points in the manifold to the vector whose defined orthogonal plane causes their images under Φ_v to be identical, Φ_v fails to be injective whenever v is in the image of g. However, g maps a $2m$-dimensional manifold into an N-dimensional manifold, so that the image of g has measure 0 whenever $N > 2m + 1$.

Now we analyze the second condition under which Φ_v might fail to be an embedding. We will analyze this condition under a specific map U, ϕ, since it is a local condition. $D\Phi_v$ will fail to be injective precisely when v is in the normalized image of $\Phi \circ \phi : \phi(U) \subset \mathbb{R}^m \to \mathbb{R}^N$, since if it is, there will be lines in M traveling along the v direction, which will then all be taken to a single point under Φ_v. Thus we have a map

$$\frac{D(\Phi \circ \phi^{-1})}{||D(\Phi \circ \phi^{-1})||} : U \times S^{m-1} \to S^{N-1}, \quad (5)$$

which has measure 0 as long as $m + (m-1) < N - 1$, which is true, since this is a less restrictive condition than $2m < N - 1$ above. Thus Φ_v fails to be an embedding only on a set of v of only measure 0, so that we can pick some v for which it is, and $N = 2n + 1$.

The Strong Whitney Embedding Theorem: Let M be a manifold of dimension m. Then there exists an embedding $f : M \to \mathbb{R}^{2m}$.

First, we note that if we allow Φ to be an immersion rather than an embedding, we can relax the stricter requirement of injectivity of the map, so that we have $N = 2m$. This results in an immersion $\Phi_v : M \to \mathbb{R}^{2m}$ with transverse self-intersections. First, we consider the more general case, where $m \geq 3$, for reasons that will shortly become clear. Recall that we have transverse self-intersections in the immersed image. A completely rigorous proof is beyond the reach of this survey, so we will merely give a sketch here. In essence, what we need to do is apply purely local moves in order to remove transverse self-intersections. We can do this by introducing another transverse intersection. This having been done, we construct a simple closed curve connecting the two, which will bound a disc in $\mathbb{R}^{2m}$, since $\mathbb{R}^{2m}$ is simply connected. Finally, we isotope the simple closed curve through itself through a one-parameter family of immersions to eliminate the singularities, resulting in an embedding.

Finally, we check all manifolds for $m \leq 2$. This is trivial for dimension 0, and the only compact manifold of dimension 1 is S^1, which also trivially embeds into $\mathbb{R}^2$. For 2-manifolds, we know that we can classify these as all being S^2, a connected sum of tori, or a connect sum of copies of $\mathbb{RP}^2$, so that it suffices to show this for S^2, T^2, and $\mathbb{RP}^2$. S^2, T^2 trivially embed into $\mathbb{R}^3$, and because $\mathbb{RP}^2$ is nonorientable, it can't be embedded in $\mathbb{R}^3$, and can in $\mathbb{R}^4$ at minimum.

Finally, we note that this boundary is sharp among linear boundaries, since, for instance, as above, $\mathbb{RP}^2$ embeds only in $\mathbb{R}^4$.

Embeddings and Knot Theory

The concept of embedding is vital in knot theory in particular, where we define knots as embeddings $f : S^1 \hookrightarrow \mathbb{R}^3$, in order to match the alternative notion of a knot as the canonical embedding $g : S^1 \to \mathbb{R}^3$ given by $t \mapsto (cos(t), sin(t), 0)$, where we recall that S^1 is defined as $\mathbb{R}/\mathbb{Z}$, which we then allow to isotope through itself until we reach another embedding.

We briefly take a different approach to showing any two embeddings of a 1-manifold into $\mathbb{R}^4$ are isotopic. Let K be a knot embedded in $\mathbb{R}^3 \subsetneq \mathbb{R}^4$ without loss of generality. Then for each pair of adjacent crossings of K in which a strand of the knot does not pass under or over the other strands of the knot, we can remove the crossings in pairs. In this way, we can reduce the knot to an unknot. Similarly, by reversing the isotopy taking any knot in $\mathbb{R}^4$ to the unknot, we can then convert this unknot to an arbitrary knot. Then since knots are the only embeddings of S^1 in $\mathbb{R}^4$, every embedding of S^1 in R^n for $n \geq 4$ must be isotopic.

Examples of Uses of Embedding

The concept of an embedding has appeared implicitly on multiple of the geometry and topology preliminary examinations, but only once recently in explicit use. From the Spring 2012 Smooth Manifolds Preliminary Examination, Question 2:

Let $P \subset \mathbb{R}^3$ be a finite set. Show that there exists a smooth embedding $f : S^2 \to \mathbb{R}^3$ such that $P \subset f(S^2)$. Then show that the corresponding statement for countably infinite sets is false.

A sketch of a proof: We know that embeddings are generic; that is, it is always possible to perturb an embedding, deforming or moving it smoothly by a small amount, and get another embedding with the same domain and codomain, so that perturbing it a finite number of times preserves the embedding. However, we can construct countably infinite sets that can't be in any embedding, since $\mathbb{Q}^3 \subset \mathbb{R}^3$ is an everywhere dense countable set. We can thus take some strict immersion, that is, an immersion that isn't an embedding, and consider the set of points in the image of the map with all rational coordinates; since the image is dense in the strict immersion, just as $\mathbb{Q}^3$ is dense in $\mathbb{R}^3$, it can't be a subset of any embedding, since such an embedding would have to contain a self-intersection, which is impossible.

Recent Developments, Open Problems, and Further Notes

A current open problem is that of which 3-manifolds can be embedded in $\mathbb{R}^4$, and of whether S^3 embedded in $\mathbb{R}^4$ will always bound a 4-ball, and in general of which m-manifolds can be embedded in $\mathbb{R}^{2m}$. However, relatively little classical work has been done in the field since around 1970, since, as previously stated, the linear boundary is a sharp one. That said, the idea of embedding has been generalized numerous times in the study of hyperbolic manifolds, braid groups, and other mathematical structures. In some sense, whenever one mathematical object can contain a copy of another, we can stretch the notion of the embedding to cover such cases, encompassing such ideas as parts of braids, subgroups, and, of course, differential-geometric embeddings into spaces that aren't Euclidean, the most interesting examples being those of complex and hyperbolic codomains.

Less recently, the Nash Embedding Theorem is a statement in Riemannian geometry analogous to the Whitney embedding theorems' role in differential geometry, saying that every Riemannian manifold can be isometrically embedded in $\mathbb{R}^{m+1}$, where the embedding is in C^1 and ϵ-close to an immersion into the same Euclidean space. In some sense this is a further refinement of the Whitney embedding theorems when restricted to extremely well-behaved manifolds, just as isometric embedding is a refinement of embedding when restricted to those same extremely well-behaved manifolds.

Relating to knot theory, a still-open problem is that of the slice-ribbon conjecture. A ribbon knot is an embedding of S^1 in $\mathbb{R}^3$ that bounds an immersion of B^2 in $\mathbb{R}^3$, that is, one which bounds an immersed disc in $\mathbb{R}^3$.

A slice knot is an embedding of S^1 in $\mathbb{R}^3$ that bounds an embedding of B^2 in $\mathbb{R}^4$, that is, it bounds an embedded disc in $\mathbb{R}^4$. The open question is, is every slice knot ribbon? We can give an elementary proof that every ribbon knot is slice: given an immersed disc in $\mathbb{R}^3$, we can use the additional dimension of freedom to isotope away all of the transverse self-intersections 4-above the hyperplane of the inclusion of $\mathbb{R}^3$ into $\mathbb{R}^4$, so that the disc is now immersed.

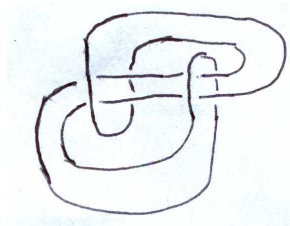

Figure 5: A ribbon knot. Note that transverse intersections. By an elementary theorem, it is also slice.

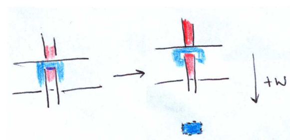

Figure 6: Smoothly isotoping away the transverse intersections of the immersed disc to embed.

On the other hand, as we can see both from an intuitive appreciation of the problem and from the idea of orthogonal projection of an embedding that we have used in the the Weak Whitney Embedding theorem, although every orthogonal projection of the embedded disc down to $\mathbb{R}^3$ results in an immersion, such projections will likely not result in embeddings. Further, the Whitney trick doesn't work in $\mathbb{R}^3, \mathbb{R}^4$, due to a lack of degrees of freedom, so we can't necessarily simply isotope the resulting self-intersections away in $\mathbb{R}^3$.

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